Large and moderate deviation principles for recursive kernel density estimators defined by stochastic approximation method

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Abstract

In this paper we prove large and moderate deviations principles for the recursive kernel estimators of a probability density function defined by the stochastic approximation algorithm introduced by Mokkadem et al. [2009. The stochastic approximation method for the estimation of a probability density. J. Statist. Plann. Inference 139, 2459-2478]. We show that the estimator constructed using the stepsize which minimize the variance of the class of the recursive estimators defined in Mokkadem et al. (2009) gives the same pointwise LDP and MDP as the Rosenblatt kernel estimator. We provide results both for the pointwise and the uniform deviations.

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1 Introduction

Let $X_1, \ldots, X_n$ be independent, identically distributed $\mathbb{R}^d$-valued random vectors, and let f denote the probability density of X_1. To construct a stochastic algorithm, which approximates the function f at a given point x, Mokkadem et al. (2009) defined an algorithm of search of the zero of the function $h\ : y \mapsto f(x) - y$. They proceed as follows: (i) they set $f_0(x) \in \mathbb{R}$; (ii) for all $n \geq 1$, they set

$$f_n(x) = f_{n-1}(x) + \gamma_n W_n(x)$$

where $W_n(x)$ is an "observation" of the function h at the point $f_{n-1}(x)$ and (γ_n) is a sequence of positive real numbers that goes to zero. To define $W_n(x)$, they follow the approach of Révész (1973, 1977) and of Tsybakov (1990), and introduced a kernel K (which is a function satisfying $\int_{\mathbb{R}^d} K(x)dx = 1$) and a bandwidth (h_n) (which is a sequence of positive real numbers that goes to zero), and they set $W_n(x) = h_n^{-d} K(h_n^{-1}[x - X_n]) - f_{n-1}(x)$. The stochastic approximation algorithm introduced in Mokkadem et al. (2009) which estimate recursively the density f at the point x is

$$f_n(x) = (1 - \gamma_n) f_{n-1}(x) + \gamma_n h_n^{-d} K\left(\frac{x - X_n}{h_n}\right). \tag{1}$$

Recently, large and moderate deviations results have been proved for the well-known nonrecursive kernel density estimator introduced by Rosenblatt (1956) (see also Parzen, 1962). The large deviations principle has been studied by Louani (1998) and Worms (2001). Gao (2003) and Mokkadem et al. (2005) extend these results and provide moderate deviations principles. The purpose of this paper is to establish large and moderate deviations principles for the recursive density estimator defined by the stochastic approximation algorithm (1).

Let us first recall that a $\mathbb{R}^m$-valued sequence $(Z_n)_{n\geq 1}$ satisfies a large deviations principle (LDP) with speed (ν_n) and good rate function I if :

1. (ν_n) is a positive sequence such that $\lim_{n\to\infty} \nu_n = \infty$;
2. $I : \mathbb{R}^m \to [0, \infty]$ has compact level sets;
3. for every borel set $B \subset \mathbb{R}^m$,

$$\begin{aligned} -\inf_{x\in \overset{\circ}{B}} I(x) &\leq \liminf_{n\to\infty} \nu_n^{-1} \log \mathbb{P}\left[Z_n \in B\right] \\ &\leq \limsup_{n\to\infty} \nu_n^{-1} \log \mathbb{P}\left[Z_n \in B\right] \leq -\inf_{x\in \overline{B}} I(x), \end{aligned}$$

where $\overset{\circ}{B}$ and $\overline{B}$ denote the interior and the closure of B respectively. Moreover, let (v_n) be a nonrandom sequence that goes to infinity; if $(v_n Z_n)$ satisfies a LDP, then (Z_n) is said to satisfy a moderate deviations principle (MDP).

The first aim of this paper is to establish pointwise LDP for the recursive kernel density estimators defined by the stochastic approximation algorithm (1). It turns out that the rate function depend on the choice of the stepsize (γ_n); In the first part of this paper we focus on the following two special cases : (1) $(\gamma_n) = (n^{-1})$ and (2) $(\gamma_n) = \left(h_n^d \left(\sum_{k=1}^n h_k^d\right)^{-1}\right)$, the first one belongs to the subclass of recursive kernel estimators which have a minimum MSE or MISE and the second choice belongs to the subclass of recursive kernel estimators which have a minimum variance (see Mokkadem et al., 2009).

We show that using the stepsize $(\gamma_n) = (n^{-1})$ and $(h_n) \equiv (cn^{-a})$ with $c > 0$ and $a \in]0, 1/d[$, the sequence $(f_n(x) - f(x))$ satisfies a LDP with speed (nh_n^d) and the rate function defined as follows:

$$\begin{cases} if \quad f(x) \neq 0, \quad I_{a,x} : t \to f(x) I_a\left(1 + \frac{t}{f(x)}\right) \\ if \quad f(x) = 0, \quad I_{a,x}(0) = 0 \quad and \quad I_{a,x}(t) = +\infty \quad for \quad t \neq 0. \end{cases} \tag{2}$$

where

$$I_a(t) = \sup_{u \in \mathbb{R}} \{ut - \psi_a(u)\}$$
$$\psi_a(u) = \int_{[0,1]\times\mathbb{R}^d} s^{-ad}\left(e^{us^{ad}K(z)} - 1\right) ds dz,$$

which is the same rate function for the LDP of the Wolverton and Wagner (1969) kernel estimator (see Mokkadem et al., 2006).

Moreover, we show that using the stepsize $(\gamma_n) = \left(h_n^d \left(\sum_{k=1}^n h_k^d\right)^{-1}\right)$ and more general bandwiths defined as $h_n = h(n)$ for all n, where h is a regularly varing function with exponent $(-a)$, $a \in]0, 1/d[$. We prove that the sequence $(f_n(x) - f(x))$ satisfies a LDP with speed (nh_n^d) and the rate function defined as follows:

$$\begin{cases} if \quad f(x) \neq 0, \quad I_x : t \to f(x) I\left(1 + \frac{t}{f(x)}\right) \\ if \quad f(x) = 0, \quad I_x(0) = 0 \quad and \quad I_x(t) = +\infty \quad for \quad t \neq 0. \end{cases} \tag{3}$$

where

$$I(t) = \sup_{u \in \mathbb{R}} \{ut - \psi(u)\}$$
$$\psi(u) = \int_{\mathbb{R}^d} \left(e^{uK(z)} - 1\right) dz,$$

which is the same rate function for the LDP of the Rosenblatt kernel estimator (see Mokkadem et al., 2005).

Our second aim is to provide pointwise MDP for the density estimator defined by the stochastic approximation algorithm (1). In this case, we consider more general stepsizes defined as $\gamma_n = \gamma(n)$ for all n, where γ is a regularly function with exponent $(-\alpha)$, $\alpha \in]1/2, 1]$. Throughout this paper we will use the following notation:

$$\xi = \lim_{n \to +\infty} (n\gamma_n)^{-1}. \tag{4}$$

For any positive sequence (v_n) satisfying

$$\lim_{n\to\infty} v_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} = 0$$

and general bandwidths (h_n), we prove that the sequence

$$v_n (f_n(x) - f(x))$$

satisfies a LDP of speed $\left(h_n^d/\left(\gamma_n v_n^2\right)\right)$ and rate function $J_{a,\alpha,x}(.)$ defined by

$$\begin{cases} if \quad f(x) \neq 0, \quad J_{a,\alpha,x} : t \to \frac{t^2(2-(\alpha-ad)\xi)}{2f(x)\int_{\mathbb{R}^d} K^2(z)dz} \\ if \quad f(x) = 0, \quad J_{a,\alpha,x}(0) = 0 \quad and \quad J_{a,\alpha,x}(t) = +\infty \quad for \quad t \neq 0. \end{cases} \tag{5}$$

Let us point out that using the stepsize $(\gamma_n) = \left(h_n^d\left(\sum_{k=1}^d h_k^d\right)^{-1}\right)$ which minimize the variance of f_n, we obtain the same rate function for the pointwise LDP and MDP as the one obtained for the Rosenblatt kernel estimator.
Finally, we give a uniform version of the previous results. More precisely, let U be a subset of $\mathbb{R}^d$; we establish large and moderate deviations principles for the sequence $\left(\sup_{x\in U}|f_n(x) - f(x)|\right)$.

2 Assumptions and main results

We define the following class of regularly varying sequences.

Definition 1. Let $\gamma \in \mathbb{R}$ and $(v_n)_{n\geq 1}$ be a nonrandom positive sequence. We say that $(v_n) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n\to+\infty} n\left[1 - \frac{v_{n-1}}{v_n}\right] = \gamma. \tag{6}$$

Condition (6) was introduced by Galambos and Seneta (1973) to define regularly varying sequences (see also Bojanic and Seneta, 1973), and by Mokkadem and Pelletier (2007) in the context of stochastic approximation algorithms. Typical sequences in $\mathcal{GS}(\gamma)$ are, for $b \in \mathbb{R}$, $n^\gamma(\log n)^b$, $n^\gamma(\log\log n)^b$, and so on.

2.1 Pointwise LDP for the density estimator defined by the stochastic approximation algorithm (1)

2.1.1 Choices of (γ_n) minimizing the MISE of f_n

It was shown in Mokkadem et al. (2009) that to minimize the MISE of f_n, the stepsize (γ_n) must be chosen in $\mathcal{GS}(-1)$ and must satisfy $\lim_{n\to\infty} n\gamma_n = 1$. The most simple example of stepsize belonging to $\mathcal{GS}(-1)$ and such that $\lim_{n\to\infty} n\gamma_n = 1$ is $(\gamma_n) = \left(n^{-1}\right)$. For this choice of stepsize, the estimator f_n defined by (1) equals the recursive kernel estimator introduced by Wolverton and Wagner (1969).

To establish pointwise LDP for f_n in this case, we need the following assumptions.

(L1) $K : \mathbb{R}^d \to \mathbb{R}$ is a bounded and integrable function satisfying $\int_{\mathbb{R}^d} K(z)\,dz = 1$, and $\lim_{\|z\|\to\infty} K(z) = 0$.

(L2) i) $(h_n) = (cn^{-a})$ with $a \in]0, 1/d[$ and $c > 0$.
ii) $(\gamma_n) = \left(n^{-1}\right)$.

The following Theorem gives the pointwise LDP for f_n in this case.

Theorem 1 (Pointwise LDP for Wolverton and Wagner estimator).
Let Assumptions $(L1)$ and $(L2)$ hold and assume that f is continuous at x. Then, the sequence $(f_n(x) - f(x))$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function defined by (2).

2.1.2 Choices of (γ_n) minimizing the variance of f_n

It was shown in Mokkadem et al. (2009) that to minimize the asymptotic variance of f_n, the stepsize (γ_n) must be chosen in $\mathcal{GS}(-1)$ and must satisfy $\lim_{n\to\infty} n\gamma_n = 1 - ad$. The most simple example of stepsize belonging to $\mathcal{GS}(-1)$ and such that $\lim_{n\to\infty} n\gamma_n = 1 - ad$ is $(\gamma_n) = \left((1-ad)\,n^{-1}\right)$, an other stepsize satisfying this conditions is $(\gamma_n) = \left(h_n^d\left(\sum_{k=1}^n h_k^d\right)^{-1}\right)$. For this last choice of stepsize, the estimator f_n defined by (1) produces the estimator considered by Deheuvels (1973) and Duflo (1997).
To establish pointwise LDP for f_n in this case, we assume that.

(L3) i) $(h_n) \in \mathcal{GS}(-a)$ with $a \in]0, 1/d[$.
$ii)$ $(\gamma_n) = \left(h_n^d \left(\sum_{k=1}^n h_k^d \right)^{-1} \right)$.

The following Theorem gives the pointwise LDP for f_n in this case.

Theorem 2 (Pointwise LDP for Deheuvels estimator).
Let Assumptions $(L1)$ and $(L3)$ hold and assume that f is continuous at x. Then, the sequence $(f_n(x) - f(x))$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function defined by (3).

2.2 Pointwise MDP for the density estimator defined by the stochastic approximation algorithm (1)

Let (v_n) be a positive sequence; we assume that

(M1) $K : \mathbb{R}^d \to \mathbb{R}$ is a continuous, bounded function satisfying $\int_{\mathbb{R}^d} K(z)\, dz = 1$, and, for all $j \in \{1, \ldots d\}$, $\int_{\mathbb{R}} z_j K(z)\, dz_j = 0$ and $\int_{\mathbb{R}^d} z_j^2 |K(z)|dz < \infty$.

(M2) $i)$ $(\gamma_n) \in \mathcal{GS}(-\alpha)$ with $\alpha \in]1/2, 1]$.
$ii)$ $(h_n) \in \mathcal{GS}(-a)$ with $a \in]0, \alpha/d[$.
$iii)$ $\lim_{n\to\infty} (n\gamma_n) \in]\min\{2a, (\alpha - ad)/2\}, \infty]$.

(M3) f is bounded, twice differentiable, and, for all $i, j \in \{1, \ldots d\}$, $\partial^2 f/\partial x_i \partial x_j$ is bounded.

(M4) $\lim_{n\to\infty} v_n = \infty$ and $\lim_{n\to\infty} \gamma_n v_n^2 / h_n^d = 0$.

The following Theorem gives the pointwise MDP for f_n.

Theorem 3 (Pointwise MDP for the recursive estimators defined by (1)).
Let Assumptions $(M1) - (M4)$ hold and assume that f is continuous at x. Then, the sequence $(f_n(x) - f(x))$ satisfies a MDP with speed $\left(h_n^d / \left(\gamma_n v_n^2\right)\right)$ and rate function $J_{a,\alpha,x}$ defined in (5).

2.3 Uniform LDP and MDP for the density estimator defined by the stochastic approximation algorithm (1)

To establish uniform large deviations principles for the density estimator defined by the stochastic approximation algorithm (1) on a bounded set, we need the following assumptions:

(U1) $i)$ For all $j \in \{1, \ldots d\}$, $\int_{\mathbb{R}} z_j K(z)\, dz_j = 0$ and $\int_{\mathbb{R}^d} z_j^2 |K(z)|dz < \infty$.
$ii)$ K is Hölder continuous.

(U2) f is bounded, twice differentiable, and, $\sup_{x\in\mathbb{R}^d} \|D^2 f(x)\| < \infty$.

(U3) $\lim_{n\to\infty} \frac{\gamma_n v_n^2 \log(1/h_n)}{h_n^d} = 0$ and $\lim_{n\to\infty} \frac{\gamma_n v_n^2 \log v_n}{h_n^d} = 0$.

Set $U \subseteq \mathbb{R}^d$; in order to state in a compact form the uniform large and moderate deviations principles for the density estimator defined by the stochastic approximation algorithm (1) on U, we set:

$$
\begin{aligned}
g_U(\delta) &= \begin{cases} \|f\|_{U,\infty} I_a \left(1 + \frac{\delta}{\|f\|_{U,\infty}}\right) & \text{when} \quad v_n \equiv 1 \quad , \quad (L1) \text{ and } (L2) \text{ hold} \\ \|f\|_{U,\infty} I \left(1 + \frac{\delta}{\|f\|_{U,\infty}}\right) & \text{when} \quad v_n \equiv 1 \quad , \quad (L1) \text{ and } (L3) \text{ hold} \\ \frac{\delta^2 (2 - (\alpha - ad)\xi)}{2\|f\|_{U,\infty} \int_{\mathbb{R}^d} K^2(z) dz} & \text{when} \quad v_n \to \infty \quad , \quad (M1) - (M4) \text{ hold} \end{cases} \\
\tilde{g}_U(\delta) &= \min\{g_U(\delta), g_U(-\delta)\}
\end{aligned}
$$

where $\|f\|_{U,\infty} = \sup_{x\in U} |f(x)|$.

Remark 1. The functions $g_U(.)$ and $\tilde{g}_U(.)$ are non-negative, continuous, increasing on $]0, +\infty[$ and decreasing on $]-\infty, 0[$, with a unique global minimum in 0 ($\tilde{g}_U(0) = g_U(0) = 0$). They are thus good rate functions (and $g_U(.)$ is strictly convex).

Theorem 4 below states uniform LDP on U in the case U is bounded, and Theorem 5 in the case U is unbounded.

Theorem 4 (Uniform deviations on a bounded set for the recursive estimator defined by (1)). Let $(U1)-(U3)$ hold. Then for any bounded subset U of $\mathbb{R}^d$ and for all $\delta > 0$,

$$\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|f_n(x) - f(x)\right| \geq \delta\right] = -\tilde{g}_U(\delta) \tag{7}$$

To establish uniform large deviations principles for the density estimator defined by the stochastic approximation algorithm (1) on an unbounded set, we need the following additionnal assumptions:

(U4) $i)$ There exists $\beta > 0$ such that $\int_{\mathbb{R}^d} \|x\|^\beta f(x)\, dx < \infty$.
$ii)$ f is uniformly continuous.

(U5) There exists $\tau > 0$ such that $z \mapsto \|z\|^\tau K(z)$ is a bounded function.

(U6) $i)$ There exists $\zeta > 0$ such that $\int_{\mathbb{R}^d} \|z\|^\zeta |K(z)|\, dz < \infty$
$ii)$ There exists $\eta > 0$ such that $z \mapsto \|z\|^\eta f(z)$ is a bounded function.

Theorem 5 (Uniform deviations on an unbounded set for the recursive estimator defined by (1)). Let $(U1)-(U6)$ hold. Then for any subset U of $\mathbb{R}^d$ and for all $\delta > 0$,

$$\begin{aligned} -\tilde{g}_U(\delta) &\leq \liminf_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|f_n(x) - f(x)\right| \geq \delta\right] \\ &\leq \limsup_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|f_n(x) - f(x)\right| \geq \delta\right] \leq -\frac{\beta}{\beta+d}\tilde{g}_U(\delta) \end{aligned}$$

The following corollary is a straightforward consequence of Theorem 5.

Corollary 1. Under the assumptions of Theorem 5, if $\int_{\mathbb{R}^d} \|x\|^\xi f(x)\, dx < \infty$ for all ξ in $\mathbb{R}$, then for any subset U of $\mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|f_n(x) - f(x)\right| \geq \delta\right] = -\tilde{g}_U(\delta) \tag{8}$$

Comment. Since the sequence $(\sup_{x\in U} |f_n(x) - f(x)|)$ is positive and since $\tilde{g}_U$ is continuous on $[0, +\infty[$, increasing and goes to infinity as $\delta \to \infty$, the application of Lemma 5 in Worms (2001) allows to deduce from (7) or (8) that $\sup_{x\in U} |f_n(x) - f(x)|$ satisfies a LDP with speed $\left(\gamma_n^{-1} h_n^d\right)$ and good rate function $\tilde{g}_U$ on $\mathbb{R}^+$.

3 Proofs

Throught this section we use the following notation:

$$\begin{aligned} \Pi_n &= \prod_{j=1}^n (1-\gamma_j), \\ Z_n(x) &= h_n^{-d} Y_n, \\ Y_n &= K\left(\frac{x - X_n}{h_n}\right) \end{aligned} \tag{9}$$

Throughout the proofs, we repeatedly apply Lemma 2 in Mokkadem et al. (2009). For the convenience of the reader, we state it now.

Lemma 1. Let $(v_n) \in \mathcal{GS}(v^*)$, $(\gamma_n) \in \mathcal{GS}(-\alpha)$, and $m > 0$ such that $m - v^*\xi > 0$ where ξ is defined in (4). We have

$$\lim_{n\to+\infty} v_n \Pi_n^m \sum_{k=1}^n \Pi_k^{-m} \frac{\gamma_k}{v_k} = \frac{1}{m - v^*\xi}.$$

Moreover, for all positive sequence (α_n) such that $\lim_{n\to+\infty} \alpha_n = 0$, and for all $\delta \in \mathbb{R}$,

$$\lim_{n\to+\infty} v_n \Pi_n^m \left[\sum_{k=1}^n \Pi_k^{-m} \frac{\gamma_k}{v_k} \alpha_k + \delta\right] = 0.$$

Noting that, in view of (1), we have

$$\begin{aligned} f_n(x) - f(x) &= (1-\gamma_n)\left(f_{n-1}(x) - f(x)\right) + \gamma_n\left(Z_n(x) - f(x)\right) \\ &= \sum_{k=1}^{n-1}\left[\prod_{j=k+1}^{n}(1-\gamma_j)\right]\gamma_k\left(Z_k(x) - f(x)\right) + \gamma_n\left(Z_n(x) - f(x)\right) + \left[\prod_{j=1}^{n}(1-\gamma_j)\right]\left(f_0(x) - f(x)\right) \\ &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\left(Z_k(x) - f(x)\right) + \Pi_n\left(f_0(x) - f(x)\right). \end{aligned}$$

It follows that

$$\mathbb{E}\left[f_n(x)\right] - f(x) = \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\left(\mathbb{E}\left[Z_k(x)\right] - f(x)\right) + \Pi_n\left(f_0(x) - f(x)\right).$$

Then, we can write that

$$\begin{aligned} f_n(x) - \mathbb{E}\left[f_n(x)\right] &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\left(Z_k(x) - \mathbb{E}\left[Z_k(x)\right]\right) \\ &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-d}\left(Y_k - \mathbb{E}\left[Y_k\right]\right) \end{aligned}$$

Let (Ψ_n) and (B_n) be the sequences defined as

$$\begin{aligned} \Psi_n(x) &= \Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k h_k^{-d}\left(Y_k - \mathbb{E}\left[Y_k\right]\right) \\ B_n(x) &= \mathbb{E}\left[f_n(x)\right] - f(x) \end{aligned}$$

We have:

$$f_n(x) - f(x) = \Psi_n(x) + B_n(x) \tag{10}$$

Theorems 1, 2, 3, 4 and 5 are consequences of (10) and the following propositions.

Proposition 1 (Pointwise LDP and MDP for (Ψ_n)).

1. Under the assumptions $(L1)$ and $(L2)$, the sequence $\left(f_n(x) - \mathbb{E}\left(f_n(x)\right)\right)$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function $I_{a,x}$.
2. Under the assumptions $(L1)$ and $(L3)$, the sequence $\left(f_n(x) - \mathbb{E}\left(f_n(x)\right)\right)$ satisfies a LDP with speed $\left(nh_n^d\right)$ and rate function I_x.
3. Under the assumptions $(M1)$–$(M4)$, the sequence $\left(v_n\Psi_n(x)\right)$ satisfies a LDP with speed $\left(h_n^d/\left(\gamma_n v_n^2\right)\right)$ and rate function $J_{a,\alpha,x}$.

Proposition 2 (Uniform LDP and MDP for (Ψ_n)).

1. Let $(U1)-(U3)$ hold. Then for any bounded subset U of $\mathbb{R}^d$ and for all $\delta > 0$,

$$\lim_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n(x)\right| \geq \delta\right] = -\tilde{g}_U(\delta)$$

2. Let $(U1)-(U6)$ hold. Then for any subset U of $\mathbb{R}^d$ and for all $\delta > 0$,

$$\begin{aligned} -\tilde{g}_U(\delta) &\leq \liminf_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n(x)\right| \geq \delta\right] \\ &\leq \limsup_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U}v_n\left|\Psi_n(x)\right| \geq \delta\right] \leq -\frac{\xi}{\xi+d}\tilde{g}_U(\delta) \end{aligned}$$

The proof of the following proposition is given in Mokkadem et al. (2009).

Proposition 3 (Pointwise and uniform convergence rate of (B_n)).
Let Assumptions $(M1) - (M3)$ hold.

1. If for all $i, j \in \{1, \dots d\}$, $\partial^2 f / \partial x_i \partial x_j$ is continuous at x. We have

   If $a \leq \alpha/(d+4)$, then

$$B_n(x) = O\left(h_n^2\right).$$

   If $a > \alpha/(d+4)$, then

$$B_n(x) = o\left(\sqrt{\gamma_n h_n^{-d}}\right).$$

2. If $(U2)$ holds, then:

   If $a \leq \alpha/(d+4)$, then

$$\sup_{x \in \mathbb{R}^d} |B_n(x)| = O\left(h_n^2\right).$$

   If $a > \alpha/(d+4)$, then

$$\sup_{x \in \mathbb{R}^d} |B_n(x)| = o\left(\sqrt{\gamma_n h_n^{-d}}\right).$$

Set $x \in \mathbb{R}^d$; since the assumptions of Theorems 1 and 2 guarantee that $\lim_{n\to\infty} B_n(x) = 0$, Theorem 1 (respectively Theorem 2) is a straightforward consequence of the application of Part 1 (respectively of Part 2) of Proposition 1. Moreover, under the assumptions of Theorem 3, we have by application of Propostion 3, $\lim_{n\to\infty} v_n B_n(x) = 0$; Theorem 3 thus straightfully follows from the application of Part 3 of Proposition 1. Finaly, Theorem 4 and 5 follows from Proposition 2 and the second part of Proposition 3. We now state a preliminary lemma, which will be used in the proof of Proposition 1.
For any $u \in \mathbb{R}$, Set

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{E}\left[\exp\left(u \frac{h_n^d}{\gamma_n v_n} \Psi_n(x)\right)\right] \\
\Lambda_x^{L,1}(u) &= f(x)\left(\psi_a(u) - u\right), \\
\Lambda_x^{L,2}(u) &= f(x)\left(\psi(u) - u\right), \\
\Lambda_x^{M}(u) &= \frac{u^2}{2\left(2 - (\alpha - ad)\xi\right)} f(x) \int_{\mathbb{R}^d} K^2(z)\, dz
\end{aligned}$$

Lemma 2. [Convergence of $\Lambda_{n,x}$]

1. (Pointwise convergence)
   If f is continuous at x, then for all $u \in \mathbb{R}$

$$\lim_{n\to\infty} \Lambda_{n,x}(u) = \Lambda_x(u) \tag{11}$$

   where

$$\Lambda_x(u) = \begin{cases} \Lambda_x^{L,1}(u) & \text{when} \quad v_n \equiv 1 \quad , \ (L1) \text{ and } (L2) \text{ hold} \\ \Lambda_x^{L,2}(u) & \text{when} \quad v_n \equiv 1 \quad , \ (L1) \text{ and } (L3) \text{ hold} \\ \Lambda_x^{M}(u) & \text{when} \quad v_n \to \infty \quad , \ (M1) - (M4) \text{ hold} \end{cases}$$

2. (Uniform convergence)
   If f is uniformly continuous, then the convergence (11) holds uniformly in $x \in U$.

Our proofs are now organized as follows: Lemma 2 is proved in Section 3.1, Proposition 1 in Section 3.4 and Proposition 2 in Section 3.3.

3.1 Proof of Lemma 2.

Set $u \in \mathbb{R}$, $u_n = u/v_n$ and $a_n = h_n^d \gamma_n^{-1}$. We have:

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{v_n^2}{a_n} \log \mathbb{E}\left[\exp\left(u_n a_n \Psi_n(x)\right)\right] \\
&= \frac{v_n^2}{a_n} \log \mathbb{E}\left[\exp\left(u_n a_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_k^{-1}\left(Y_k - \mathbb{E}\left[Y_k\right]\right)\right)\right] \\
&= \frac{v_n^2}{a_n} \sum_{k=1}^n \log \mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right)\right] - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_k^{-1} \mathbb{E}\left[Y_k\right]
\end{aligned}$$

By Taylor expansion, there exists $c_{k,n}$ between 1 and $\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right)\right]$ such that

$$\log \mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right)\right] = \mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right] - \frac{1}{2 c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right]\right)^2$$

and $\Lambda_{n,x}$ can be rewriten as

$$\begin{aligned}
\Lambda_{n,x}(u) = {} & \frac{v_n^2}{a_n} \sum_{k=1}^n \mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right] - \frac{v_n^2}{2 a_n} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right]\right)^2 \\
& - u v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} a_k^{-1} \mathbb{E}\left[Y_k\right] \qquad (12)
\end{aligned}$$

First case: $v_n \to \infty$**.** A Taylor's expansion implies the existence of $c'_{k,n}$ between 0 and $u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k$ such that

$$\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right] = u_n \frac{a_n \Pi_n}{a_k \Pi_k} \mathbb{E}\left[Y_k\right] + \frac{1}{2}\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k}\right)^2 \mathbb{E}\left[Y_k^2\right] + \frac{1}{6}\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k}\right)^3 \mathbb{E}\left[Y_k^3 e^{c'_{k,n}}\right]$$

Therefore,

$$\begin{aligned}
\Lambda_{n,x}(u) &= \frac{1}{2} u^2 a_n \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_k^{-2} \mathbb{E}\left[Y_k^2\right] + \frac{1}{6} u^2 u_n a_n^2 \Pi_n^3 \sum_{k=1}^n \Pi_k^{-3} a_k^{-3} \mathbb{E}\left[Y_k^3 e^{c'_{k,n}}\right] \\
&\quad - \frac{v_n^2}{2 a_n} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right]\right)^2 \\
&= \frac{1}{2} f(x) u^2 a_n \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_k^{-1} \gamma_k \int_{\mathbb{R}^d} K^2(z)\, dz + R_{n,x}^{(1)}(u) + R_{n,x}^{(2)}(u) \qquad (13)
\end{aligned}$$

with

$$\begin{aligned}
R_{n,x}^{(1)}(u) &= \frac{1}{2} u^2 a_n \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_k^{-1} \gamma_k \int_{\mathbb{R}^d} K^2(z)\left[f(x - z h_k) - f(x)\right] dz \\
R_{n,x}^{(2)}(u) &= \frac{1}{6} \frac{u^3}{v_n} a_n^2 \Pi_n^3 \sum_{k=1}^n \Pi_k^{-3} a_k^{-3} \mathbb{E}\left[Y_k^3 e^{c'_{k,n}}\right] - \frac{v_n^2}{2 a_n} \sum_{k=1}^n \frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n \frac{a_n \Pi_n}{a_k \Pi_k} Y_k\right) - 1\right]\right)^2
\end{aligned}$$

Since f is continuous, we have $\lim_{k \to \infty} \left|f(x - z h_k) - f(x)\right| = 0$, and thus, by the dominated convergence theorem, $(M1)$ implies that

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} K^2(z) \left|f(x - z h_k) - f(x)\right| dz = 0.$$

Since $(a_n) \in \mathcal{GS}(\alpha - ad)$, and $\lim_{n \to \infty}(n \gamma_n) > (\alpha - ad)/2$. Lemma 1 then ensures that

$$a_n \Pi_n^2 \sum_{k=1}^n \Pi_k^{-2} a_k^{-1} \gamma_k = \frac{1}{(2 - (\alpha - ad)\xi)} + o(1), \qquad (14)$$

it follows that $\lim_{n\to\infty}\left|R_{n,x}^{(1)}(u)\right|=0$.
Moreover, in view of (9), we have $|Y_k|\leq\|K\|_\infty$, then

$$\begin{aligned} c'_{k,n} &\leq \left|u_n\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right| \\ &\leq |u_n|\,\|K\|_\infty \end{aligned}\tag{15}$$

Noting that $\mathbb{E}|Y_k|^3\leq h_k^d\|f\|_\infty\int_{\mathbb{R}^d}\left|K^3(z)\right|dz$. Hence, using Lemma 1 and (15), there exists a positive constant c_1 such that, for n large enough,

$$\left|\frac{u^3}{v_n}a_n^2\Pi_n^3\sum_{k=1}^n\Pi_k^{-3}a_k^{-3}\mathbb{E}\left[Y_k^3e^{c'_{k,n}}\right]\right| \leq c_1e^{|u_n|\|K\|_\infty}\frac{u^3}{v_n}\|f\|_\infty\int_{\mathbb{R}^d}\left|K^3(z)\right|dz \tag{16}$$

which goes to 0 as $n\to\infty$ since $v_n\to\infty$.
Moreover, Lemma 1 ensures that

$$\begin{aligned} &\left|\frac{v_n^2}{2a_n}\sum_{k=1}^n\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right)-1\right]\right)^2\right| \\ &\quad\leq \frac{v_n^2}{2a_n}\sum_{k=1}^n\left(\mathbb{E}\left[\exp\left(u_n\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right)-1\right]\right)^2 \\ &\quad\leq \frac{u^2}{2}\|f\|_\infty^2a_n\Pi_n^2\sum_{k=1}^n\Pi_k^{-2}a_k^{-1}\gamma_kh_k^d+o\left(a_n\Pi_n^2\sum_{k=1}^n\Pi_k^{-2}a_k^{-1}\gamma_kh_k^d\right) \\ &\quad= o(1) \end{aligned}\tag{17}$$

The combination of (16) and (17) ensures that $\lim_{n\to\infty}\left|R_{n,x}^{(2)}(u)\right|=0$. Then, we obtain from (13) and (14), $\lim_{n\to\infty}\Lambda_{n,x}(u)=\Lambda_x^M(u)$.

Second case: $(v_n)\equiv 1$**.** It follows from (12) that

$$\begin{aligned} \Lambda_{n,x}(u) &= \frac{1}{a_n}\sum_{k=1}^n\mathbb{E}\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right)-1\right]-\frac{1}{2a_n}\sum_{k=1}^n\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right)-1\right]\right)^2 \\ &\quad -u\Pi_n\sum_{k=1}^n\Pi_k^{-1}a_k^{-1}\mathbb{E}[Y_k] \\ &= \frac{1}{a_n}\sum_{k=1}^nh_k^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}K(z)\right)-1\right]f(x)\,dz-u\Pi_n\sum_{k=1}^n\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}K(z)f(x)\,dz \\ &\quad -R_{n,x}^{(3)}(u)+R_{n,x}^{(4)}(u) \\ &= f(x)\frac{1}{a_n}\sum_{k=1}^nh_k^d\left[\int_{\mathbb{R}^d}\left(\exp\left(uV_{n,k}K(z)\right)-1\right)-uV_{n,k}K(z)\right]dz \\ &\quad -R_{n,x}^{(3)}(u)+R_{n,x}^{(4)}(u) \end{aligned}\tag{18}$$

with

$$\begin{aligned} V_{n,k} &= \frac{a_n\Pi_n}{a_k\Pi_k} \\ R_{n,x}^{(3)}(u) &= \frac{1}{2a_n}\sum_{k=1}^n\frac{1}{c_{k,n}^2}\left(\mathbb{E}\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}Y_k\right)-1\right]\right)^2 \\ R_{n,x}^{(4)}(u) &= \frac{1}{a_n}\sum_{k=1}^nh_k^d\int_{\mathbb{R}^d}\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}K(z)\right)-1\right]\left[f(x-zh_k)-f(x)\right]dz \\ &\quad -u\Pi_n\sum_{k=1}^n\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}K(z)\left[f(x-zh_k)-f(x)\right]dz. \end{aligned}$$

It follows from (17), that $\lim_{n\to\infty}\left|R_{n,x}^{(3)}(u)\right| = 0$.
Since $|e^t - 1| \le |t|\, e^{|t|}$, we have

$$\begin{aligned}
\left|R_{n,x}^{(4)}(u)\right| &\le \frac{1}{a_n}\sum_{k=1}^{n} h_k^d \int_{\mathbb{R}^d}\left|\left[\exp\left(u\frac{a_n\Pi_n}{a_k\Pi_k}K(z)\right)-1\right][f(x-zh_k)-f(x)]\right| dz \\
&\quad + |u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}|K(z)|\,|f(x-zh_k)-f(x)|\,dz \\
&\le |u|\, e^{|u|\|K\|_\infty}\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}|K(z)|\,|f(x-zh_k)-f(x)|\,dz \\
&\quad + |u|\,\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}|K(z)|\,|f(x-zh_k)-f(x)|\,dz \\
&\le |u|\left(e^{|u|\|K\|_\infty}+1\right)\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\int_{\mathbb{R}^d}|K(z)|\,|f(x-zh_k)-f(x)|\,dz
\end{aligned}$$

In view of Lemma 1 the sequence $\left(\Pi_n\sum_{k=1}^{n}\Pi_k^{-1}\gamma_k\right)$ is bounded, then, the dominated convergence theorem ensures that $\lim_{n\to\infty} R_{n,x}^{(4)}(u) = 0$.
In the case f is uniformly continuous, set $\varepsilon > 0$ and let $M > 0$ such that $2\|f\|_\infty \int_{\|z\|\le M}|K(z)|\,dz \le \varepsilon/2$. We need to prove that for n sufficiently large

$$\sup_{x\in\mathbb{R}^d}\int_{\|z\|\le M}|K(z)|\,|f(x-zh_k)-f(x)|\,dz \le \varepsilon/2$$

which is a straightforward consequence of the uniform continuity of f.

Then, it follows from (18), that

$$\lim_{n\to\infty}\Lambda_{n,x}(u) = \lim_{n\to\infty} f(x)\frac{\gamma_n}{h_n^d}\sum_{k=1}^{n}h_k^d\int_{\mathbb{R}^d}\left[\left(\exp\left(uV_{n,k}K(z)\right)-1\right)-uV_{n,k}K(z)\right]dz \tag{19}$$

In the case when $(v_n)\equiv 1$, $(L1)$ and $(L2)$ hold
We have

$$\begin{aligned}
\frac{\Pi_n}{\Pi_k} &= \prod_{j=k+1}^{n}(1-\gamma_j) \\
&= \frac{k}{n},
\end{aligned}$$

then,

$$\begin{aligned}
V_{n,k} &= \frac{a_n\Pi_n}{a_k\Pi_k} \\
&= \left(\frac{k}{n}\right)^{ad}.
\end{aligned}$$

Consequently, it follows from (19) and from some analysis considerations that

$$\begin{aligned}
\lim_{n\to\infty}\Lambda_{n,x}(u) &= f(x)\int_{\mathbb{R}^d}\left[\int_0^1 s^{-ad}\left(\exp\left(us^{ad}K(z)\right)-1-us^{ad}K(z)\right)ds\right]dz \\
&= \Lambda_x^{L,1}(u)
\end{aligned}$$

In the case when $(v_n) \equiv 1$, $(L1)$ and $(L3)$ hold
We have

$$\begin{aligned}
\frac{\Pi_n}{\Pi_k} &= \prod_{j=k+1}^{n} (1-\gamma_j) \\
&= \prod_{j=k+1}^{n} \left(1 - \frac{h_j^d}{\sum_{l=1}^{j} h_l^d}\right) \\
&= \prod_{j=k+1}^{n} \frac{\sum_{l=1}^{j-1} h_l^d}{\sum_{l=1}^{j} h_l^d} \\
&= \frac{\sum_{l=1}^{k} h_l^d}{\sum_{l=1}^{n} h_l^d} \\
&= \frac{\sum_{l=1}^{k} h_l^d}{h_k^d} \frac{h_k^d}{h_n^d} \frac{h_n^d}{\sum_{l=1}^{n} h_l^d} \\
&= \frac{\gamma_n}{\gamma_k} \frac{h_k^d}{h_n^d},
\end{aligned}$$

then,

$$V_{n,k} = 1.$$

Consequently, it follows from (19) that

$$\begin{aligned}
\lim_{n\to\infty} \Lambda_{n,x}(u) &= f(x) \int_{\mathbb{R}^d} \left[(\exp(uK(z)) - 1) - uK(z) \right] dz \\
&= \Lambda_x^{L,2}(u)
\end{aligned}$$

and thus Lemma 1 is proved.

3.2 Proof of Proposition 1

To prove Proposition 1, we apply Proposition 1 in Mokkadem et al. (2006), Lemma 2 and the following result (see Puhalskii, 1994).

Lemma 3. Let (Z_n) be a sequence of real random variables, (ν_n) a positive sequence satisfying $\lim_{n\to\infty} \nu_n = +\infty$, and suppose that there exists some convex non-negative function Γ defined on $\mathbb{R}$ such that

$$\forall u \in \mathbb{R}, \lim_{n\to\infty} \frac{1}{\nu_n} \log \mathbb{E}\left[\exp(u\nu_n Z_n)\right] = \Gamma(u).$$

If the Legendre function Γ^* of Γ is a strictly convex function, then the sequence (Z_n) satisfies a `LDP` of speed (ν_n) and good rate fonction Γ^*.

In our framework, when $v_n \equiv 1$ and $\gamma_n = n^{-1}$, we take $Z_n = f_n(x) - \mathbb{E}(f_n(x))$, $\nu_n = nh_n^d$ with $h_n = cn^{-a}$ where $a \in]0, 1/d[$ and $\Gamma = \Lambda_x^{L,1}$. In this case, the Legendre transform of $\Gamma = \Lambda_x^{L,1}$ is the rate function $I_{a,x} : t \to f(x) I_a\left(\frac{t}{f(x)} + 1\right)$ which is strictly convex by Proposition 1 in Mokkadem et al. (2006). Farther, when $v_n \equiv 1$ and $\gamma_n = h_n^d \left(\sum_{k=1}^{n} h_k^d\right)^{-1}$, we take $Z_n = f_n(x) - \mathbb{E}(f_n(x))$, $\nu_n = nh_n^d$ with $h_n \in \mathcal{GS}(-a)$ where $a \in]0, 1/d[$ and $\Gamma = \Lambda_x^{L,2}$. In this case, the Legendre transform of $\Gamma = \Lambda_x^{L,2}$ is the rate function $I_x : t \to f(x) I\left(\frac{t}{f(x)} + 1\right)$ which is strictly convex by Proposition 1 in Mokkadem et al. (2005). Otherwise, when, $v_n \to \infty$, we take $Z_n = v_n(f_n(x) - \mathbb{E}(f_n(x)))$, $\nu_n = h_n^d / \left(\gamma_n v_n^2\right)$ and $\Gamma = \Lambda_x^M$; Γ^* is then the quadratic rate function $J_{a,\alpha,x}$ defined in (5) and thus Proposition 1 follows.

3.3 Proof of Proposition 2

In order to prove Proposition 2, we first establish some lemmas.

Lemma 4. Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be the function defined for $\delta > 0$ as

$$\phi(\delta) \quad = \quad \begin{cases} (\psi_a')^{-1}\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right) & \text{when} \quad v_n \equiv 1 \quad , \quad (L1) \text{ and } (L2) \text{ hold} \\ (\psi')^{-1}\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right) & \text{when} \quad v_n \equiv 1 \quad , \quad (L1) \text{ and } (L3) \text{ hold} \\ \frac{\delta(2-(\alpha-ad)\xi)}{\|f\|_{U,\infty}\int_{\mathbb{R}^d} K^2(z)dz} & \text{when} \quad v_n \to \infty \quad , \quad (M1)-(M4) \text{ hold} \end{cases}$$

1. $\sup_{u\in\mathbb{R}}\{u\delta - \sup_{x\in U}\Lambda_x(u)\}$ equals $g_U(\delta)$ and is achieved for $u = \phi(\delta) > 0$.
2. $\sup_{u\in\mathbb{R}}\{-u\delta - \sup_{x\in U}\Lambda_x(u)\}$ equals $g_U(\delta)$ and is achieved for $u = \phi(-\delta) < 0$.

Proof of Lemma 4 . We just prove the first part, the proof of the second part one being similar.

- First case : $v_n \equiv 1$, $(L1)$ and $(L2)$ hold.
  Since $e^t \geq 1+t$, for all t, we have $\psi_a(u) \geq u$ and therefore,

  $$\begin{aligned} u\delta - \sup_{x\in U}\Lambda_x(u) &= u\delta - \|f\|_{U,\infty}(\psi_a(u)-u) \\ &= \|f\|_{U,\infty}\left[u\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right)-\psi_a(u)\right] \end{aligned}$$

  The function $u \mapsto u\delta - \sup_{x\in U}\Lambda_x(u)$ has second derivative $-\|f\|_{U,\infty}\psi_a''(u) < 0$ and thus it has a unique maximum achieved for

  $$u_0 = (\psi_a')^{-1}\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right)$$

  Now, since ψ_a' is increasing and since $\psi_a'(0) = 1$, we deduce that $u_0 > 0$.

- Second case : $v_n \equiv 1$, $(L1)$ and $(L3)$ hold.
  Since $e^t \geq 1+t$, for all t, we have $\psi(u) \geq u$ and therefore,

  $$\begin{aligned} u\delta - \sup_{x\in U}\Lambda_x(u) &= u\delta - \|f\|_{U,\infty}(\psi(u)-u) \\ &= \|f\|_{U,\infty}\left[u\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right)-\psi(u)\right] \end{aligned}$$

  The function $u \mapsto u\delta - \sup_{x\in U}\Lambda_x(u)$ has second derivative $-\|f\|_{U,\infty}\psi''(u) < 0$ and thus it has a unique maximum achieved for

  $$u_0 = (\psi')^{-1}\left(1+\frac{\delta}{\|f\|_{U,\infty}}\right)$$

  Now, since ψ' is increasing and since $\psi'(0) = 1$, we deduce that $u_0 > 0$.

- Third case $v_n \to \infty$ and $(M2)$ holds. In this case, we have

  $$u\delta - \sup_{x\in U}\Lambda_x(u) \quad = \quad u\delta - \frac{u^2}{2(2-(\alpha-ad)\xi)}\|f\|_{U,\infty}\int_{\mathbb{R}^d} K^2(z)\,dz.$$

  In view of the assumption $(M2)$, we have $\xi^{-1} > (\alpha - ad)/2$, then the function $u \mapsto u\delta - \sup_{x\in U}\Lambda_x(u)$ has second derivative $-\frac{1}{(2-(\alpha-ad)\xi)}\|f\|_{U,\infty}\int_{\mathbb{R}^d} Kr(z)\,dz < 0$ and thus it has a unique maximum achieved for

  $$u_0 = \frac{\delta(2-(\alpha-ad)\xi)}{\|f\|_{U,\infty}\int_{\mathbb{R}^d} K^2(z)\,dz} > 0$$

Lemma 5.

- In the case when $(v_n) \equiv 1$ and $(\gamma_n) = \left(n^{-1}\right)$, let $(L1)$ and $(L2)$ hold;
- In the case when $(v_n) \equiv 1$ and $(\gamma_n) = \left(h_n^d\left(\sum_{k=1}^n h_k^d\right)^{-1}\right)$, let $(L1)$ and $(L3)$ hold;

- In the case when $v_n \to \infty$, let $(M1)-(M4)$ hold.
  Then for any $\delta > 0$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \sup_{x\in U} \mathbb{P}\left[v_n \Psi_n(x) \geq \delta\right] &= -g_U(\delta) \\
\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \sup_{x\in U} \mathbb{P}\left[v_n \Psi_n(x) \leq -\delta\right] &= -g_U(-\delta) \\
\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \sup_{x\in U} \mathbb{P}\left[v_n \left|\Psi_n(x)\right| \leq -\delta\right] &= -\tilde{g}_U(-\delta)
\end{aligned}$$

Proof of Lemma 5. The proof of Lemma 5 is similar to the proof of Lemma 4 in Mokkadem et al. (2006).

Lemma 6. Let Assumptions $(U1)-(U3)$ hold and assume that either $(v_n) \equiv 1$ or $(U4)$ holds.

1. If U is a bounded set, then for any $\delta > 0$, we have

$$\lim_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U} v_n \left|\Psi_n(x)\right|\right] \leq -\tilde{g}_U(\delta)$$

2. If U is an unbounded set, then, for any $b > 0$ and $\delta > 0$,

$$\limsup_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n \left|\Psi_n(x)\right|\right] \leq db - \tilde{g}_U(\delta)$$

   where $w_n = \exp\left(\frac{bh_n^d}{\gamma_n v_n^2}\right)$.

Proof of Lemma 6. Set $\rho \in]0, \delta[$, let β denote the Hölder order of K, and $\|K\|_H$ its corresponding Hölder norm. Set $w_n = \exp\left(\frac{bh_n^d}{\gamma_n v_n^2}\right)$ and

$$R_n = \left(\frac{\rho}{2\|K\|_H v_n \Pi_n \sum_{k=1}^n \Pi_k^{-1} \gamma_k h_k^{-(d+\beta)}}\right)^{\frac{1}{\beta}}$$

We begin with the proof of the second part of Lemma 6. There exist $N'(n)$ points of $\mathbb{R}^d$, $y_1^{(n)}, y_2^{(n)}, \ldots, y_{N'(n)}^{(n)}$ such that the ball $\left\{x \in \mathbb{R}^d; \|x\| \leq w_n\right\}$ can covered by the $N'(n)$ balls $B_i^{(n)} = \left\{x \in \mathbb{R}^d; \|x - y_i^{(n)}\| \leq R_n\right\}$ and such that $N'(n) \leq 2\left(\frac{2w_n}{R_n}\right)^d$. Considering only the $N(n)$ balls that intersect $\{x \in U; \|x\| \leq w_n\}$, we can write

$$\{x \in U; \|x\| \leq w_n\} \subset \cup_{i=1}^{N(n)} B_i^{(n)}.$$

For each $i \in \{1, \ldots, N(n)\}$, set $x_i^{(n)} \in B_i^{(n)} \cap U$. We then have:

$$\begin{aligned}
\mathbb{P}\left[\sup_{x\in U, \|x\|\leq w_n} v_n \left|\Psi_n(x)\right| \geq \delta\right] &\leq \sum_{i=1}^{N(n)} \mathbb{P}\left[\sup_{x\in B_i^{(n)}} v_n \left|\Psi_n(x)\right| \geq \delta\right] \\
&\leq N(n) \max_{1\leq i\leq N(n)} \mathbb{P}\left[\sup_{x\in B_i^{(n)}} v_n \left|\Psi_n(x)\right| \geq \delta\right].
\end{aligned}$$

Now, for any $i \in \{1, \ldots, N(n)\}$ and any $x \in B_i^{(n)}$,

$$\begin{aligned}
v_n |\Psi_n| &\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \\
&\quad + v_n \Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-d} \left| K\left(\frac{x - X_k}{h_k}\right) - K\left(\frac{x_i^{(n)} - X_k}{h_k}\right) \right| \\
&\quad + v_n \Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-d} \mathbb{E} \left| K\left(\frac{x - X_k}{h_k}\right) - K\left(\frac{x_i^{(n)} - X_k}{h_k}\right) \right| \\
&\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| + 2 v_n \|K\|_H \Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-d} \left(\frac{\|x - x_i^{(n)}\|}{h_k}\right)^{\beta} \\
&\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| + 2 v_n \|K\|_H \Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-(d+\beta)} R_n^{\beta} \\
&\leq v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| + \rho
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\mathbb{P}\left[\sup_{x \in U, \|x\| \leq w_n} v_n |\Psi_n(x)| \geq \delta\right] &\leq N(n) \max_{1 \leq i \leq N(n)} \mathbb{P}\left[v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \geq \delta - \rho\right] \\
&\leq N(n) \sup_{x \in U} \mathbb{P}\left[v_n \left|\Psi_n\left(x_i^{(n)}\right)\right| \geq \delta - \rho\right]
\end{aligned}$$

Further, by definition of $N(n)$ and w_n, we have

$$\log N(n) \leq \log N'(n) \leq db \frac{h_n^d}{\gamma_n v_n^2} + (d+1) \log 2 - d \log R_n$$

and

$$\frac{\gamma_n v_n^2}{h_n^d} \log R_n = \frac{\gamma_n v_n^2}{\beta h_n^d} \left[\log \rho - \log\left(2\|K\|_H\right) - \log v_n - \log\left(\Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-(d+\beta)}\right)\right].$$

Moreover, we have $\left(h_n^{(d+\beta)}\right) \in \mathcal{GS}(-a(d+\beta))$. Lemma 1 ensures that

$$\Pi_n \sum_{k=1}^{n} \Pi_k^{-1} \gamma_k h_k^{-(d+\beta)} = O\left(h_n^{-(d+\beta)}\right),$$

then, in view of $(U3)$, we have

$$\limsup_{n \to \infty} \frac{\gamma_n v_n^2}{h_n^d} \log N(n) \leq db \tag{20}$$

The application of Lemma 5 then yiels

$$\begin{aligned}
\limsup_{n \to \infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\sup_{x \in U, \|x\| \leq w_n} v_n |\Psi_n(x)| \geq \delta\right] &\leq \limsup_{n \to \infty} \frac{\gamma_n v_n^2}{h_n^d} \log N(n) - \tilde{g}_U(\delta - \rho) \\
&\leq db - \tilde{g}_U(\delta - \rho).
\end{aligned}$$

Since the inequality holds for any $\rho \in]0, \delta[$, part 2 of Lemma 6 thus follows from the continuity of $\tilde{g}_U$.

Let us now consider part 1 of Lemma 6. This part is proved by following the same steps as for part 2, except that the number $N(n)$ of balls covering U is at most the integer part of $(\Delta/R_n)^d$, where Δ denotes the diameter of $\overline{U}$. Relation (20) then becomes

$$\limsup_{n \to \infty} \frac{\gamma_n v_n^2}{h_n^d} \log R_n \leq 0$$

and Lemma 6 is proved.

Lemma 7. Let $(U1)\,i)$, $(M2)$ and $(U6)\,i)$ hold. Assume that either $(v_n) \equiv 1$ or $(U3)$ and $(U6)\,ii)$ hold. Moreover assume that f is continuous. For any $b>0$ if we set $w_n = \exp\left(\frac{bh_n^d}{\gamma_n v_n^2}\right)$ then, for any $\rho > 0$, we have, for n large enough,

$$\sup_{x\in U, \|x\|\geq w_n} v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k h_k^{-d}\left|\mathbb{E}\left[K\left(\frac{x-X_k}{h_k}\right)\right]\right| \leq \rho$$

Proof of Lemma 7. We have

$$v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k h_k^{-d}\mathbb{E}\left[K\left(\frac{x-X_k}{h_k}\right)\right] = v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\mathbb{R}^d} K(z)\, f(x-zh_k)\, dz. \tag{21}$$

First, Lemma 1, ensures that

$$\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k = 1 + o(1). \tag{22}$$

Set $\rho > 0$. In the case $(v_n) \equiv 1$, we set M such that $\|f_\infty\| \int_{\|z\|>M} |K(z)|\, dz \leq \rho/2$; it follows from (22) that

$$\begin{aligned}
&v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k h_k^{-d}\left|\mathbb{E}\left[K\left(\frac{x-X_k}{h_k}\right)\right]\right| \\
&\leq \frac{\rho}{2} + f(x)\int_{\|z\|\leq M} |K(z)|\, dz \\
&+\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\|z\|>M} |K(z)|\, |f(x-zh_k) - f(x)|\, dz.
\end{aligned}$$

Lemma 7 then follows from the fact that f fulfills $(U6)\,ii)$. As matter of fact, this conditions implies that $\lim_{\|x\|\to\infty, x\in\overline{U}} f(x) = 0$ and that the third term in the right-hand-side of the previous inequality goes to 0 as $n\to\infty$ (by the dominated convergence).
Let us now assume that $\lim_{n\to\infty} v_n = \infty$; relation (21) can be rewritten as

$$\begin{aligned}
v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k h_k^{-d}\mathbb{E}\left[K\left(\frac{x-X_k}{h_k}\right)\right] &= v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\|z\|\leq w_n/2} K(z)\, f(x-zh_k)\, dz \\
&+ v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\|z\|\geq w_n/2} K(z)\, f(x-zh_k)\, dz.
\end{aligned}$$

First, since $\|x\| \geq w_n$ and $\|z\| \leq w_n/2$, we have

$$\begin{aligned}
\|x - zh_k\| &\geq w_n(1-h_i/2) \\
&\geq w_n/2 \quad \text{for } n \text{ large enough.}
\end{aligned}$$

Moreover, in view of assumptions $(U3)$, for all $\xi > 0$,

$$\lim_{n\to\infty} \frac{v_n}{w_n^\xi} = \lim_{n\to\infty} \exp\left\{-\xi b\frac{h_n^d}{\gamma_n v_n^2}\left(1 - \frac{v_n^2\log v_n}{\xi b h_n^d}\right)\right\} = 0. \tag{23}$$

Set $M_f = \sup_{x\in\mathbb{R}^d} \|x\|^\eta f(x)$. Assumption $(U6)\,ii)$ and equations (22), (23) implie that, for n sufficiently large,

$$\begin{aligned}
&\sup_{\|x\|\geq w_n} v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\|z\|\leq w_n/2} |K(z)\, f(x-zh_k)|\, dz \\
&\leq M_f \sup_{\|x\|\geq w_n} v_n\Pi_n \sum_{k=1}^n \Pi_k^{-1}\gamma_k \int_{\|z\|\leq w_n/2} |K(z)|\, \|x-zh_k\|^{-\eta} dz \\
&\leq 2^\eta M_f \frac{v_n}{w_n^\eta}\int_{\mathbb{R}^d} |K(z)|\, dz \\
&\leq \frac{\rho}{2}.
\end{aligned}$$

Moreover, in view of $(U3)$, $(U6)\,i)$ and (22), (23), for n sufficiently large,

$$\begin{aligned}
&\sup_{\|x\|\geq w_n} v_n \Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k \int_{\|z\|>w_n/2} |K(z)\, f(x-zh_k)|\, dz\\
&\leq 2^{\zeta} M_f \frac{v_n}{w_n^{\zeta}} \int_{\|z\|>w_n/2} \|z\|^{\zeta} |K(z)|\, dz\\
&\leq \frac{\rho}{2}.
\end{aligned}$$

This concludes the proof of Lemma 7. Since K is a bounded function that vanishes at infinity, we have $\lim_{\|x\|\to\infty} |\Psi_n(x)| = 0$ for every $n \geq 1$. Moreover, since K is assumed to be continuous, Ψ_n is continuous, and this ensures the existence of a random variable s_n such that

$$|\Psi_n(s_n)| = \sup_{x\in U} |\Psi_n(x)|.$$

Lemma 8.
Let Assumptions $(U1)-(U3)$, $(U4)\,ii)$ and $(U5)$ hold. Suppose either $(v_n)\equiv 1$ or $(H6)$ hold. For any $b>0$, set $w_n = \exp\left(b\frac{h_n^d}{\gamma_n v_n^2}\right)$; for any $\delta>0$, we have

$$\limsup_{n\to\infty} \frac{\gamma_n v_n^2}{h_n^d} \log \mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad |\Psi_n(s_n)| \geq \delta\right] \quad \leq \quad -b\beta \tag{24}$$

Proof of Lemma 8. We first note that $s_n \in \overline{U}$ and therefore

$$\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-d} K\left(\frac{s_n - X_k}{h_k}\right)\right|\\
&\qquad +v_n \mathbb{E}\left|\Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-d} K\left(\frac{s_n - X_k}{h_k}\right)\right| \geq \delta\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad v_n \Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-d} \left|K\left(\frac{s_n - X_k}{h_k}\right)\right|\\
&\qquad - \sup_{\|x\|\geq w_n, x\in\overline{U}} v_n \Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-d} \mathbb{E}\left|K\left(\frac{s_n - X_k}{h_k}\right)\right| \geq \delta.
\end{aligned}$$

Set $\rho \in]0,\delta[$; the application of Lemma 7 ensures that, for n large enough,

$$\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad v_n \left|\Pi_n \sum_{k=1}^{n} \Pi_k^{-1}\gamma_k h_k^{-d} K\left(\frac{s_n - X_k}{h_k}\right)\right| \geq \delta - \rho.
\end{aligned}$$

Set $\kappa = \sup_{x\in\mathbb{R}^d} \|x\|^{\gamma} |K(x)|$ (see Assumption $(U5)$). We obtain, for n sufficiently large,

$$\begin{aligned}
&\|s_n\| \geq w_n \quad \text{and} \quad v_n |\Psi_n(s_n)| \geq \delta\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1,\ldots,n\} \quad \text{such that} \quad \frac{v_n}{h_k^d}\left|K\left(\frac{s_n - X_k}{h_k}\right)\right| \geq \delta - \rho\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1,\ldots,n\} \quad \text{such that} \quad \kappa h_k^{\gamma} \geq \frac{h_k^d}{v_n}\|s_n - X_k\|^{\gamma}(\delta-\rho)\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1,\ldots,n\} \quad \text{such that} \quad \left|\|s_n\| - \|X_k\|\right| \leq \left[\frac{\kappa v_n h_k^{\gamma-d}}{\delta-\rho}\right]^{\frac{1}{\gamma}}\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1,\ldots,n\} \quad \text{such that} \quad \|X_k\| \leq \|s_n\| - \left[\frac{\kappa v_n h_k^{\gamma-d}}{\delta-\rho}\right]^{\frac{1}{\gamma}}\\
&\quad \Rightarrow \|s_n\| \geq w_n \quad \text{and} \quad \exists k \in \{1,\ldots,n\} \quad \text{such that} \quad \|X_k\| \leq w_n(1-u_{n,k}) \quad \text{with}\\
&\qquad u_{n,k} = w_n^{-1} v_n^{\frac{1}{\gamma}} h_k^{\frac{\gamma-d}{\gamma}} \left(\frac{\kappa}{\delta-\rho}\right)^{\frac{1}{\gamma}}.
\end{aligned}$$

Moreover, we can write $u_{n,k}$ as

$$u_{n,k} = \exp\left(-b\frac{h_n^d}{\gamma_n v_n^2}\left[1 - \frac{1}{b\gamma}\frac{\gamma_n v_n^2 \log v_n}{h_n^d} - \frac{\gamma - d}{b\gamma}\frac{\gamma_n v_n^2 \log(h_k)}{h_n^d}\right]\right)\left(\frac{\kappa}{\delta - \rho}\right)^{\frac{1}{\gamma}}$$

and assumption $(U3)$ ensure that $\lim_{n\to\infty} u_{n,k} = 0$, it then follows that $1 - u_{n,k} > 0$ for n sufficiently large; therefore we can deduce that (see Assumption $(U4)\,i)$):

$$\begin{aligned}
\mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n\left|\Psi_n(s_n)\right| \geq \delta\right] &\leq \sum_{i=1}^{n}\mathbb{P}\left[\|X_k\|^\beta \geq w_n^\beta\left(1 - u_{n,k}\right)^\beta\right] \\
&\leq \sum_{i=1}^{n}\mathbb{E}\left(\|X_k\|^\beta\right) w_n^{-\beta}\left(1 - u_{n,k}\right)^{-\beta} \\
&\leq n\mathbb{E}\left(\|X_1\|^\beta\right) w_n^{-\beta}\max_{1\leq k\leq n}\left(1 - u_{n,k}\right)^{-\beta}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n\left|\Psi_n(s_n)\right| \geq \delta\right] \\
&\quad \leq \frac{\gamma_n v_n^2}{h_n^d}\left[\log n + \log\mathbb{E}\left(\|X_1\|^\beta\right) - b\beta\frac{h_n^d}{\gamma_n v_n^2} - \beta\log\max_{1\leq k\leq n}\left(1 - u_{n,k}\right)\right],
\end{aligned}$$

and, thanks to assumptions $(U3)$, it follows that

$$\limsup_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\|s_n\| \geq w_n \quad \text{and} \quad v_n\left|\Psi_n(s_n)\right| \geq \delta\right] \quad \leq \quad -b\beta,$$

which concludes the proof of Lemma 8.

3.4 Proof of Proposition 2

Let us at first note that the lower bound

$$\liminf_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U} v_n\left|\Psi_n(x)\right| \geq \delta\right] \geq -\tilde{g}_U(\delta) \tag{25}$$

follows from the application of Proposition 1 at a point $x_0 \in \overline{U}$ such that $f(x_0) = \|f\|_{U,\infty}$.
In the case U is bounded, Proposition 2 is thus a straightforward consequence of (25) and the first part of Lemma 6. Let us now consider the case U is unbounded.
Set $\delta > 0$ and, for any $b > 0$ set $w_n = \exp\left(b\frac{h_n^d}{\gamma_n v_n^2}\right)$. Since, by definition of s_n,

$$\begin{aligned}
&\mathbb{P}\left[\sup_{x\in U} v_n\left|\Psi_n(x)\right| \geq \delta\right] \\
&\quad \leq \mathbb{P}\left[\sup_{x\in U,\|x\|\leq w_n} v_n\left|\Psi_n(x)\right| \geq \delta\right] + \mathbb{P}\left[\|s_n\| \geq w_n \text{ and } v_n\left|\Psi_n(x)\right| \geq \delta\right],
\end{aligned}$$

it follows from Lemmas 6 and 8 that

$$\limsup_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U} v_n\left|\Psi_n(x)\right| \geq \delta\right] \quad \leq \quad \max\left\{-b\beta; db - \tilde{g}_U(\delta)\right\}$$

and consequently

$$\limsup_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U} v_n\left|\Psi_n(x)\right| \geq \delta\right] \quad \leq \quad \inf_{b>0}\max\left\{-b\beta; db - \tilde{g}_U(\delta)\right\}.$$

Since the infimum in the right-hand-side of the previous bound is achieved for $b = \tilde{g}_U(\delta)/(\beta + b)$ and equals $-\beta\tilde{g}_U/(\beta + d)$, we obtain the upper bound

$$\limsup_{n\to\infty}\frac{\gamma_n v_n^2}{h_n^d}\log\mathbb{P}\left[\sup_{x\in U} v_n\left|\Psi_n(x)\right| \geq \delta\right] \quad \leq \quad -\frac{\beta}{\beta + d}\tilde{g}_U(\delta)$$

which concludes the proof of Proposition 2.

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